UNIVERSITY OF CALIFORNIA

INSTITUTE OF GEOPHYSICS

Final Report

PROPAGATION OF SOUND IN A RARIFIED MAXWELLIAN GAS

Contract Nonr-233-(18)

LOS ANGELES 24, CALIFORNIA

University of California
Institute of Geophysics

Final Report

PROPAGATION OF SOUND IN A RaRIFIED MAXWELLIAN GAS

Contract Nonr-233(18)

University of California
Institute of Geophysics

Final Report
Contract Nonr-233(18)
December 23, 1953

PROPAGATION OF SOUND IN A RARIFIED MAXWELLIAN GAS

By

C. L. Pekeris
Head of the Department of Applied Mathematics
The Weizmann Institute in Rehovot, Israel

PROPAGATION OF SOUND IN A RARIFIED MAXWELLIAN GAS

By
C. L. Pekeris

I. INTRODUCTION

In this investigation a study was made of the dispersion and attenuation of sound in a monoatomic gas, when the density of the gas is so low, that the mean free path becomes comparable to, and eventually exceeds, the wavelength of sound. Under these conditions, also, the frequency of sound f becomes comparable to, and eventually exceeds, the mean collision frequency f_a between molecules. In helium at atmospheric pressure the required frequency of sound would be greater than 10^9 cps. Working with a frequency of 1 mc/sec, and Helium at pressures of 1 mm and less, M. Greenspan[1] succeeded in measuring the dispersion and attenuation down to a rarefaction where the collision frequency was equal to the frequency of sound ($R = 0.2$ in Figure 1.). The experiments of Greenspan extend beyond the range of validity of the Stokes-Navier hydrodynamic equation of a viscous gas. The latter equations can be relied upon only up to terms in the first power in the parameter (L/λ), where λ denotes the wavelength of sound, and L the mean free path. Similarly, the Burnett[2] hydrodynamic equations are accurate only up to the second power in (L/λ). The interpretation of the new data on the propagation of sound in helium must therefore be based on a solution of the complete equation of transfer of Boltzmann. Indeed, this is the first instance where the Boltzmann theory can be tested in a case which cannot otherwise be treated by hydrodynamic theory.

II. THE BOLTZMANN EQUATION OF TRANSFER

Let $f(x,y,z,u,v,w)$ dxdydzdudvdw denote the number of molecules in the element of volume dxdydz which have velocities in the range u and $u + du$, v and $v + dv$, w and $w + dw$. Boltzmann's equation expresses the rate of change of f due to collisions between molecules, and to the action of external forces on the molecules. When there are no external forces acting, and for the case of a plane sound wave propagating in the z direction, Boltzmann's equation takes the form

$$\partial f/\partial t + w\partial f/\partial z = \iiint du_1 dv_1 dw_1 \iint d\epsilon \sin\Theta d\Theta \; gI(g,\Theta)(f'f_1'-ff_1). \qquad (1)$$

Here the unprimed f denotes the distribution function before collision, and f' denotes the distribution function after collision; g denotes the relative velocity of the colliding molecules; $I(g,\Theta)$, the collision crossection corresponding to the turning of the relative velocity into the solid angle $\sin\Theta d\Theta d\epsilon$. The term $f'f_1'$ gives the rate of increase of f due to collisions in which the velocity vector of the molecule after collision enters the element dudvdw, while the term ff_1 represents the rate of decrease of f due to collisions in which the velocity vector of the molecule is in the element dudvdw before collision. The four velocity vectors of the two colliding molecules before and after collision are connected by the relations expressing the conservation of energy and momentum in the collision, and the law of force of interaction between molecules.

Boltzmann published his equation of transfer in 1872. By 1899 he despaired of obtaining a general solution. The problem was then taken up by Chapman and Enskog, and later by Burnett. These authors succeeded in deriving expressions for the transport coefficients under the limiting conditions when $(l/\lambda)\ll 1$. One reason for the complexity of the analysis is

the non-linear character of the equation. This difficulty does not arise
in the case of propagation of a sound wave of infinitesimal amplitudes,
when equation (1) can be linearised. We let

$$f = F(1 + h), \tag{2}$$

where F denotes the Maxwell distribution function

$$F = n(m/2\pi kT)^{3/2} \exp(-mc^2/2kT), \tag{3}$$

and n represents the number-density of the molecules. On neglecting
powers in h higher than the first, equation (1) takes the form

$$\partial h/\partial t + w \partial h/\partial r = \iiint du_1 dv_1 dw_1 \iint d\epsilon \sin\theta d\theta g I(g,\theta) F_1(h' + h_1' - h - h_1) = J(h). \tag{4}$$

This equation has recently been studied by Wang Chang and Uhlenbeck[3].
Their method is to develop h in terms of the eigenfunctions of the integral
equation

$$\lambda_i h_i = J(h_i). \tag{5}$$

Adopting a system of spherical coordinates with the polar axis in the
s-direction, they put

$$h = \sum_n h_n(r) P_n(\cos\theta). \tag{6}$$

Unfortunately, there are an infinite set of eigenfunctions $h_n^{(k)}(r)$ for
each n, so that h has to be expressed in terms of a double infinity of
eigenfunctions. They treat a Maxwellian gas, in which the interaction
between a pair of molecules is a repulsive force varying as the inverse
fifth power of the distance between the molecules. They expressed the phase
velocity and attenuation coefficient as a power series in the parameter
(L/λ), and have derived the coefficients up to the fourth power. Since
these coefficients increase rapidly, the formulas can be used only for
small values of (L/λ). Very likely, the radius of convergence of the power
series development is small, so that the method is not adequate for the in-
terpretation of the new experimental results, where $(L/\lambda) \simeq 1$.

The method which we have used to determine the propagation constants (phase velocity and attenuation coefficient) from the secular determinant, is to work successively with determinants of order 5, 8, 12, and finally 20. For each order of the determinant, the propagation constants were solved from the polynomial of the same degree representing the determinant. No expansion of the roots in powers of (L/λ) was resorted to, but the roots were solved for numerically from the polynomial. It was hoped that the convergence of the results could be judged from the degree of coincidence of the roots obtained from the determinants of successive orders. The results are shown in Figures 1 and 2.

III. RESULTS

Assuming a propagation factor for $\underline{h}$ of the form

$$\exp(i\omega t - i\sigma_1 z - \sigma_2 z). \tag{7}$$

the secular determinant yields values for

$$\mho = (V_0 \sigma/\omega) = A - iB, \tag{8}$$

where V_0 denotes the phase velocity at zero frequency. The phase velocity V, and the attenuation coefficient σ_2 are then obtained from

$$V/V_0 = 1/A, \tag{9}$$

$$\sigma_2 = B(\omega/V_0) = (2\pi/\lambda)(V/V_0)B. \tag{10}$$

The results are shown in Figure 1., where

$$R = V_0^2 \rho/\omega\mu = (2/3\pi)(f_c/f), \tag{11}$$

ρ denoting the density, μ the coefficient of viscosity, and f_c the collision frequency. R is proportional to (λ/L), the ratio of the wavelength of sound λ to the mean free path $\underline{L}$. $\underline{A}$ is proportional then to the reciprocal of the phase velocity, and $\underline{B}$ is proportional to the coefficient of absorption per wavelength.

-4-

It would appear from Figure 1 that with the determinant or order 20, the computed values for the propagation constants can be relied upon for R greater than about 3, and perhaps even for a somewhat smaller value of R. The writer expects to solve in the near future the determinant of order 30.

IV. THE SECOND MODE

Figure 2 shows the results obtained from the determinant of order 8. Here a difficulty was encountered when solving the determinant, due to the closeness of the roots of the first and second modes. The integral equation (4) gives not one propagation constant, but, very likely, and infinite number. This means that a given source of excitation placed in the gas will excite many sound waves, each propagating with a different sound velocity and with a different coefficient of attenuation. For zero frequency of sound, the attenuation of the higher modes is so much larger than that of the first mode that one does not observe them. When, however, the frequency of sound becomes comparable with the collision frequency between molecules, the attenuation in the first mode becomes large, and it may even increase above the attenuation in the second mode. Under such conditions, the mode of highest amplitude reaching a point distant several wavelengths from the source will be the second, and not the first. Such a situation results from a solution of the eighth order determinant, as shown in Figure 2. The attenuation coefficient of the first mode, shown by the curve marked BI, is less than the attenuation coefficient of the second mode BII only for $R > 1.5$. For $R < 1.5$, the second mode is the less attenuated one, and an observer studying the least attenuated wave would be determining BII and AII, rather than BI and AI.

V. FURTHER WORK

A. The results presented in this report are of a preliminary nature, pending a complete check on all the numerical work, to be undertaken shortly.

B. Should the results obtained from the determinant of order 30, and from higher orders, yield a converging theoretical solution for the propagation constants which differ substantially from the experimental values obtained for Helium, it would follow that the approximation of the real Helium gas by a Maxwellian one is not correct. This in turn would imply that the propagation constants in a gas under conditions of extreme rarefaction offer a new means of determining the law of interaction between molecules.

C. The propagation constants for the second mode will be solved from the 20-th and from the 30-th order determinants in order to determine, in the first instance, whether its attenuation does not become less than that of the first mode for small values of R.

D. Should that prove to be the case, it will be of interest to attempt to detect experimentally the existence of the second mode, as well
   to determine
as / its propagation constants. It will also be possible to study theoretically the nature of this mode, since, along with the propagation constants, one obtains from the secular determinant also the distribution function h in (2), from which all the dynamical properties, such as for instance the entropy transport, can be determined. We shall not speculate at this point on the possible connection between the second mode of sound waves in an extremely rarified gas and the "second sound" found in liquid Helium II.

E. An attempt will be made to obtain asymptotic solutions for the propagation constants for small values of R.

F. The analysis will be applied to a gas-model consisting of molecules of rigid spheres.

REFERENCES

1. M. Greenspan, Jr. Ac. Soc. Am. 22, 568 (1950)

2. D. Burnett, Proc. Lond. Math. Soc. 39, 385 (1935),
   Ibid, 40, 382 (1935)

3. C. S. Wang Chang and G. E. Uhlenbeck, Contract N6onr-23222, October 1952
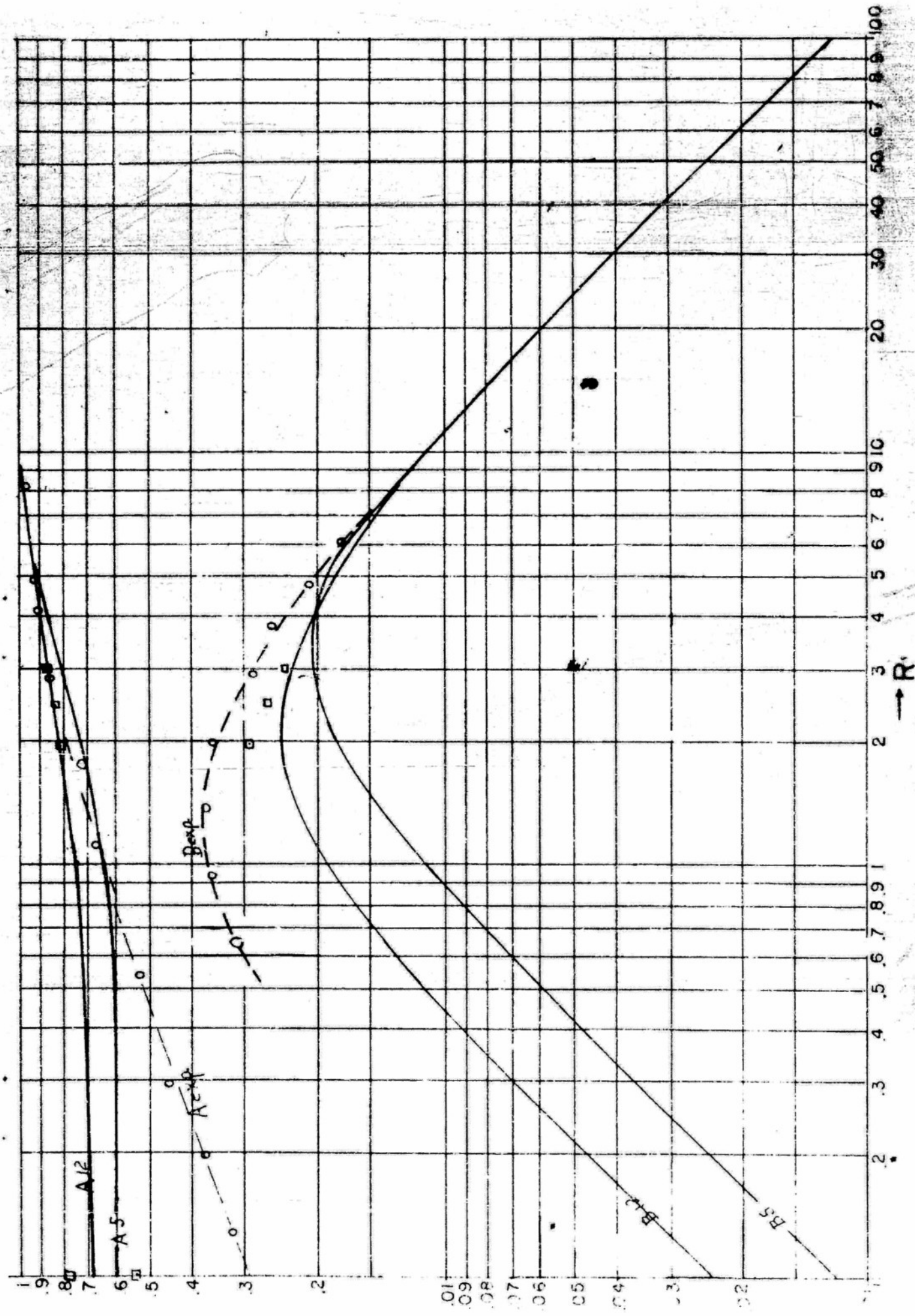
Figure 1. Propagation characteristics of sound in a Maxwellian gas. Propagation factor $\exp(i\omega t - i\sigma_1 s - \sigma_2 s)$. V = phase velocity, V_0 = phase velocity at zero frequency, $V = (V_0/A)$; $\sigma_2 = (2\pi/\lambda)(V/V_0)B$; $R = V_0^2 \rho/\omega\mu = (2/3\pi)(f_c/f)$, f_c = collision frequency, ρ = density, μ = coefficient of viscosity. The numbers on the curves designate the order of the determinant used. $\square$ = points from determinant of order 20. The experimental points were obtained by M. Greenspan for Helium (Jr. Ac. Soc. Am. 22, 568, 1950).
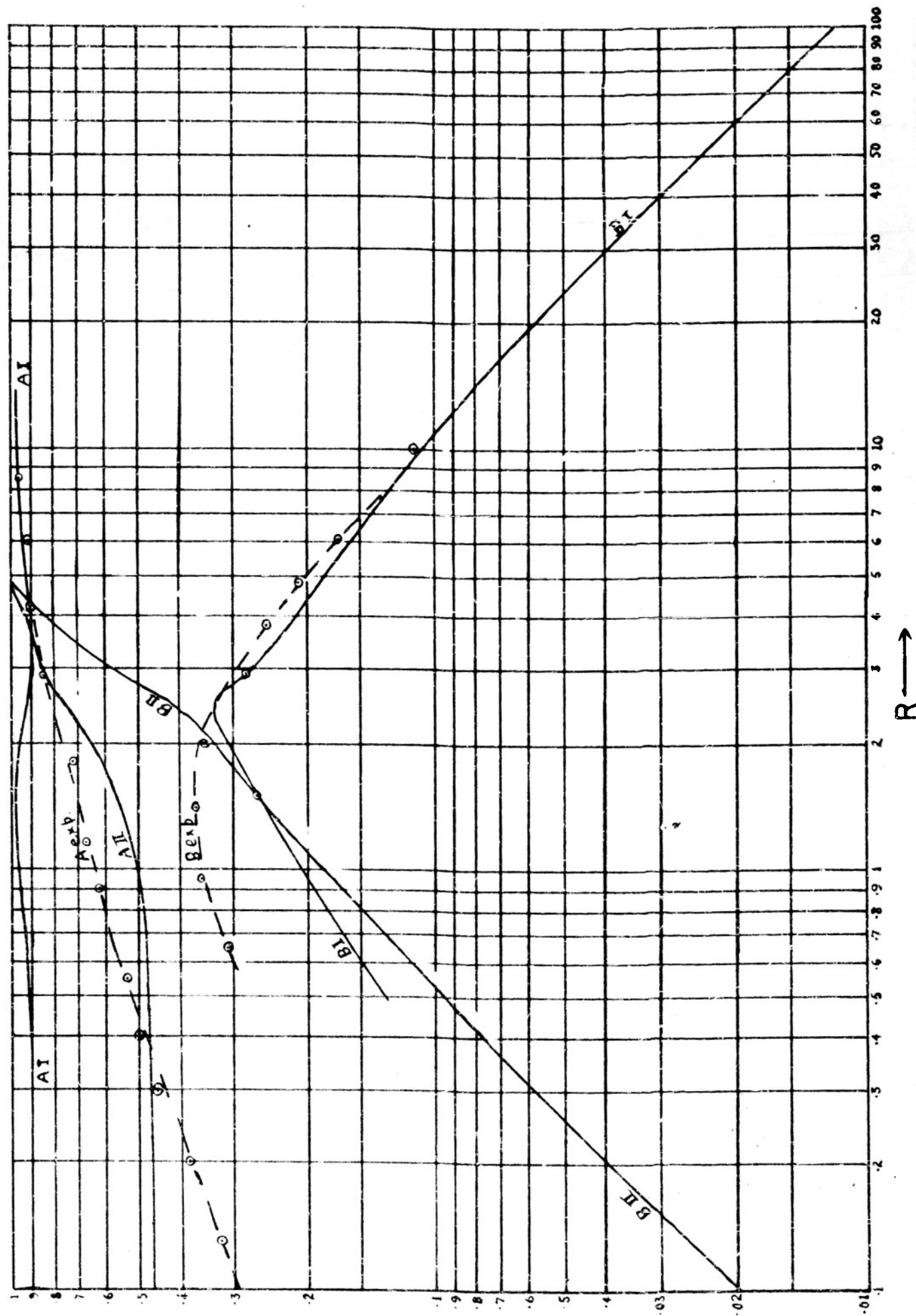
Figure 2. Propagation characteristics of the first mode (I) and of the second mode (II), as obtained from the determinant of eighth order.